

# SUPPORTING THE INTRODUCTION TO FORMAL PROOF

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*In this study, a tool that worked to support teachers with the introduction to formal proof in geometry is discussed. The tool helped teachers navigate the “shallow end” of proof. More specifically, the tool was shown to support teachers with introducing and scaffolding proof. Findings from this study suggest that the tool may be useful for supporting formal reasoning in geometry as well as other areas.*

## INTRODUCTION

Considering the teachers' role in navigating the proof terrain, Herbst (2002) conducted an analysis of what is involved when teachers attempt to engage students in the production of a proof. He argued that alternative ways of engaging students in proving must be found if proving is to play, in the classroom, the same instrumental role for knowing mathematics that it plays in the discipline. Thinking about possible instructional alternatives for the reform-oriented classroom as an opportunity, Herbst (2002) stated: “The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving” (p. 200). A primary goal of this paper is to describe and discuss the reported benefits of a tool that was developed in a research study whose aim was to better understand the challenges teachers faced when teaching proof in geometry. Following Smith and Southerland (2007), here “tool” references a teaching tool or guide that was used to help teachers envision a new way of teaching, in this case, mathematical proof. The research question addressed in this paper is: How can the mathematical proof tool (MPT) serve as a guide to support teachers' work of introducing proof in secondary geometry?

## THEORETICAL PERSPECTIVE

Past research has shown that students have difficulty with proof at various levels in many parts of the world (Knipping, 2004). The finding that most U.S. students are not developing through the van Hiele levels at all (Fuys, Geddes, & Tischler, 1988), is problematic because it implies that students enter high school unprepared for the formal deduction required in many geometry courses (Clements, 2003). This is important because students must understand geometric ideas in the middle grades in order to be successful in subsequent mathematics experiences (Sinclair, Pimm, & Skelin, 2012), including secondary level geometry. Thus, there is an obvious need for this curricular gap to be bridged. However, some secondary teachers have claimed that they do not have strategies for teaching proof and even expressed the belief that you cannot teach someone how to develop a proof (Cirillo, 2011). This belief may be the

reason that geometry is often thought of as the most difficult portion of school mathematics (Knuth, 2002).

Much of Herbst and colleagues' work has focused on classroom interactions and proving in geometry at the secondary level (Herbst & Brach, 2006; Herbst et al., 2009). For example, Herbst et al. (2009) described instances of student engagement with proof in various geometry courses in a high school. Through this work they unearthed a system of norms that appear to regulate the activity of "doing proofs" in geometry class. The authors contended that a collection of actions related to filling in the two-column form are regulated by norms that express how labor is divided between teacher and students and how time is organized as far as sequence and duration of events. They argued that despite the superficially different episodes in which doing proofs were observed, there were deep similarities among those events. The first 5 of 25 norms reported by Herbst et al. (2009) are listed below:

...producing a proof, consists of (1) writing a sequence of steps (each of which consists of a "statement" and "reason"), where (2) the first statement is the assertion of one or more "given" properties of a geometric figure, (3) each other statement asserts a fact about a specific figure using a diagrammatic register and (4) the last step is the assertion of a property identified earlier as the "prove"; during which (5) each of those asserted statements are tracked on a diagram by way of standard marks ...(pp. 254-255)

This model of the instructional situation of doing proofs in terms of a system of norms is helpful to those who wish to investigate what it might mean to create a different place for proof in geometry classrooms (Herbst et al., 2009).

The documentation of classroom norms is relevant here because it provides a frame for examining the alternative practices supported by the tool used by the teachers in this study. This study builds on the work of Herbst and colleagues by examining possibilities outside of these normative practices. It also takes seriously the call to bridge the curriculum gap by supporting students' development through the use of a teaching tool that has the potential to lead to new norms in geometry classrooms.

## **METHODS**

To learn more about the challenges that teachers face when cultivating formal proof in their classrooms, a three-year study that made use of qualitative methods of inquiry, was designed. For the larger study, five teachers who had between one and ten years of experience with teaching proof in geometry were recruited. Baseline data, collected in Fall, 2010, included two non-consecutive weeks of classroom observations in one target classroom of each teacher. Beginning Spring, 2011, 20 professional development (PD) sessions were designed and implemented to attend to and reconsider the ways in which the study teachers taught proof. These sessions took place over the course of a year. In Fall, 2011 and 2012, additional data were collected to observe and understand changes made to the introduction and teaching of proof in geometry. Interviews designed to help the researcher better understand the data and the teachers' evolving beliefs about teaching proof were also conducted.

## Data and Analysis

This paper draws on a subset of the teachers and the data described above. For this study, interview transcripts from two teachers' data sets were transcribed and analyzed. This includes a total of 4-5 interviews with each teacher, comprising a total of 3-3.5 hours per teacher spread across the three years of the project. Interviews were coded for instances where the teachers discussed how engagement with the PD and the tool influenced their practice. Data from two classroom episodes are also presented. The teaching episodes were purposefully selected because it was from these two classroom lessons that the idea to develop the tool grew. Last, a written curriculum developed by the two teachers over the second and third years of the project was analyzed. Together, this collection of data allows me to describe how the tool was developed and used over time, given the limited space provided here.

## Setting

Participants for this study include Mike and Seth (pseudonyms) who, at the onset of the study, had eight and five years of mathematics teaching experience, respectively. Mike had previously taught a high school geometry course every year since he began teaching, while Seth had only taught the geometry course once. Mike and Seth taught in a private, all-boys school with a racially diverse population and small class sizes (14-17 students). During Year 1, they taught from a conventional geometry textbook, teaching Euclidean geometry proof primarily over the course of the first semester.

## FINDINGS AND DISCUSSION

The findings in this study are explored through three data sources. Two excerpts from Mike's Year 1 baseline classroom data are presented. I then describe the Mathematical Proof Tool and explore its use in the classrooms through examples from the curriculum developed by Mike and Seth. Interview data is also included.

### The "Shallow End" of the Proof Pool

In the first year of the study, project teachers were asked to invite the research team in when they first introduced formal proof. Before beginning the first proof, Mike said the following to the students:

Here we go. So proofs are tough. You know one thing about proofs is, there's no easy way. There's no way to do it. There's no shallow end. You can't like wade into the proof pool. You gotta kind of jump right in the deep end with these tough ones. (11/2/10)

To this introduction, a student responded, "I would drown." The next day, Mike began the lesson by explaining how difficult proofs are:

These proofs are really hard and I think I said last time a couple things. One, there's no real easy way to start proofs. It's not like algebra where you could start with easy problems and work to more difficult problems and then do really challenging problems. The proofs start, and they are immediately difficult and they are immediately unlike anything that you have ever seen before and that's okay. Alright, so you'll learn how to do 'em by sort of trying them. (11/2/10)

These two examples suggest that Mike seemed to hold similar beliefs to those of the teacher in Cirillo's (2011) study of a beginning teacher learning to teach proof in geometry. Like, Matt, despite his eight years of experience teaching geometry, Mike's introduction to proof provides evidence that he was at a loss when it came to scaffolding the introduction of proof.

### **The Development of the Mathematical Proof Tool**

There were four important findings from the first year of data collection across all five project teachers. First, teachers did not understand all that was involved in teaching students how to develop and write proofs. Second, as was demonstrated, teachers did not know how to scaffold the introduction to proof. Third, teachers thought that the only way to teach proof was through show-and-tell. Last, a set of ideas that were implicitly taught during these show-and-tell presentations were found in the analysis of classroom observations. In particular, students must learn the following simultaneously: (a) postulates, definitions, and theorems; (b) how to *use* definitions to draw conclusions (c) how to work with diagrams (i.e., what can and cannot be assumed); (d) a variety of sub-arguments and negotiated classroom norms for writing them up; and (e) how sub-arguments come together to construct the larger argument. It was through these observations in conjunction with the consideration of Mike's claim that there is no shallow end to proof that the Mathematical Proof Tool (MPT) was developed. Based on the shallow end proof pool metaphor suggested by Mike, I hypothesized that perhaps there was a set of competencies that students needed in order to develop proofs that could be ramped up over time. The PD sessions and subsequent observations gave me a way to test that hypothesis.

In Spring, 2011, the group of teachers and the research team began meeting for PD sessions. In these sessions, the teachers participated in the following activities: discussing research and practitioner articles on proof and geometry, reflecting on practice through writing and watching teaching videos, participating in PD on classroom discourse, and considering alternative teaching approaches. After learning about van Hiele levels and coming to believe that their students were not ready to engage in proof, the MPT became a major focus of alternative teaching approaches.

The tool began as sample alternative tasks and evolved into the tool that is shown in Table 1. The MPT works as an instructional guide to support teachers by offering pedagogical content knowledge that breaks down the practice of proving. It unpacks the sub-goals of proof and identifies competencies that occur frequently and are necessary to make a lesson focused on proof go well. By the third year of the study, project teachers were using the MPT as a planning guide to make sure that they were addressing each of the sub-goals, and providing learning activities that would foster the competencies in their students. In the paragraphs that follow, I briefly describe the sub-goals and include examples of each. Examples come from the written curriculum that Mike and Seth developed around the sub-goals.

Sub-Goals	Description	Competencies
Understanding Mathematical Objects and Mathematical Notation	This sub-goal connects a definition and notation to a particular instance of that object.	1) Communicating a mathematical object by making use of spoken or written text
		2) Communicating or reading a mathematical object by making use of diagrams. Sometimes notation is used to mark these diagrams
		3) Communicating or reading a mathematical object by making use of symbolic notation
		4) Determining examples and non-examples
Understanding the Nature of Definitions	This sub-goal highlights the nature of definitions, their logical structure, how they are written, and how they are used.	1) Writing a “good” definition (includes necessary and sufficient properties)
		2) Knowing definitions are not unique
		3) Understanding how to write definitions as biconditionals
		4) Knowing you cannot prove a definition
Drawing Conclusions and Developing Conjectures	This sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a diagram.	1) Understanding what can and cannot be assumed from a diagram and recognizing that sometimes diagrams can be misleading
		2) Knowing when and how definitions can be used to draw a conclusion from a statement about a mathematical object
		3) Using combinations of postulates, definitions, and theorems to draw valid conclusions from some given information
		4) Developing conjectures that could be used to prove or disprove a mathematical statement where part of the process is making, testing, and refining conjectures as one works
Sub-arguments	This sub-goal presents the idea that there are common short sequences of statements and reasons that are used frequently in proofs and that these pieces may appear relatively unchanged from one proof to the next.	1) Recognizing a sub-argument as a branch of proof and how it fits into the proof
		2) Understanding what valid conclusions can be drawn from a given statement and how those make a sub-argument (e.g., knowing some commonly occurring sub-arguments)
		3) Understanding how to write a sub-argument using acceptable notation and language (often negotiated with the teacher)
Understanding Theorems	This sub-goal highlights the nature of theorems, their structure, and how they are used.	1) If applicable, marking a diagram that satisfies a hypothesis
		2) Interpreting a theorem statement to determine the hypotheses and conclusion
		3) Rewriting a theorem written in words into symbols and vice versa
		4) Understanding that a theorem is not a theorem until it has been proved (using definitions, postulates, or previously proved theorems, lemmas, and propositions) and that one cannot use the conclusions of the theorem itself to prove the conclusions of that theorem (i.e., avoiding circular reasoning)
		5) Understanding that theorems are mathematical statements that are only sometimes biconditionals
		6) Determining the theorem proved when presented with a proof
		7) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement and the connection between laws of logic and the hypothesis and conclusion of a mathematical statement

Table 1: Mathematical Proof Tool (MPT)

The first sub-goal, *Understanding Mathematical Objects and Mathematical Notation*, supports students in working with commonly used terms in geometry, for example, angle bisectors. Students need to know particular definitions since these (along with theorems and postulates) are what make up the substance of a proof. Understanding Mathematical Objects connects a definition and notation to a particular instance of that object. Mike and Seth made use of this sub-goal early and often in their first unit on



definitions and constructions. For example, students were asked if it is possible to draw a picture in which  $\overline{DF}$  bisects  $\overline{PO}$  but  $\overline{PO}$  does not bisect  $\overline{DF}$ . Students were expected to explain their answers. Students also worked with compasses, constructing medians and perpendicular bisectors, for example.

The second sub-goal, *Understanding the Nature of Definitions*, highlights the nature of definitions, their logical structure, how they are written, and how they are used. An example from the curriculum was: “Write the two conditional statements that comprise the biconditional: *Two angles are complementary if and only if their measures sum to 90 [degrees].*” Similarly, another problem asked the students to write out the complete statement in words: “Isosceles triangle  $\leftrightarrow 2 \cong$  sides.”

The *Drawing Conclusions and Developing Conjectures* sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a given diagram. This sub-goal is useful, for instance, in helping students understand what you can and cannot assume from a diagram. For example, you can assume vertical angles, but you cannot assume perpendicular lines. A benefit of explicitly attending to this sub-goal is that it helps teachers correct common errors students tend to make regarding the conclusions they draw from the given information before they begin developing formal proofs. An example of this sub-goal is provided in Figure 1.

The *Sub-arguments* sub-goal presents the idea that there are common short sequences of statements and reasons that are frequently used in proofs and that these pieces may appear relatively unchanged from one proof to the next. An example of a common sub-argument is a proof of the proposition: If lines are perpendicular, then congruent angles are formed. In the teachers’ curriculum, after reviewing some common sub-arguments, students were asked to complete sub-arguments such as the one in Figure 2, justifying each claim with a reason.

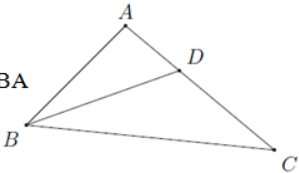
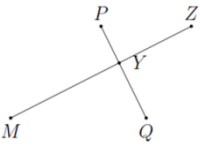
<p>Consider the problems below. In each one, draw a valid conclusion. Be sure to justify your conclusion. Marking the diagram appropriately is the best way to understand the given information.</p> <p>Given:  <math>\overline{BD}</math> bisects <math>\angle CBA</math></p> 	 <div style="margin-top: 10px;"> <div style="border: 1px solid black; padding: 5px; display: inline-block; text-align: center;"> <math>\overline{MZ} \perp \overline{PQ}</math>              Given         </div> <div style="margin-top: 10px;"> <div style="border: 1px solid black; padding: 5px; display: inline-block; text-align: center;"> <math>\angle PYZ</math> and <math>\angle QYZ</math> are rt. <math>\angle</math>'s              ( )         </div> <div style="margin-top: 10px;"> <div style="border: 1px solid black; padding: 5px; display: inline-block; text-align: center;"> <math>\angle QYZ \cong \angle PYZ</math>              ( )         </div> </div> </div> </div>
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Figure 1: Sub-arguments Example

Figure 2: Drawing Conclusions Example

Last, the *Understanding Theorems* sub-goal highlights the nature of theorems, their structure, and how they are used. For example, rather than always providing students with a diagram, a given statement, and a conclusion to prove, students are asked to set up the proofs themselves. A sample problem from Mike and Seth’s curriculum was as follows: “Set up the following statement to be proved: *If a figure is a parallelogram, then its opposite sides are congruent.*”

## Teachers' Reactions to the MPT Implementation

After the spring and summer PD that followed the baseline data collection in Year 1, Mike explained how one of the readings (see Cirillo, 2009) influenced his thinking about how he taught proof in geometry:

One of the readings...was Ten Things I Wish I Knew, and I was like van Hiele levels, give me a break. I don't wish I knew that. But, I actually wish I knew that [laughing]. So one of my 'aha' moments is that we have to adjust the curriculum, adjust our approach so that we're communicating with our students. (Mike, 8/23/11)

In an interview during Year 2, Mike discussed the types of tasks he engaged his students with through the new curriculum that he started developing that semester (Seth later partnered with Mike in teaching with the new curriculum). Mike described the *Understanding Mathematical Objects* example provided above and said:

I never would've asked this before. But just getting at the idea, you gotta look at what's bisecting what. There's a subject and an object there. Here was bisects but is not perpendicular. Perpendicular but does not bisect. Perpendicular and bisects. So does such a thing even exist? Oh perpendicular bisector. So now you come back here and construct a perpendicular bisector. (Mike, 10/6/11)

During an interview at the conclusion of Year 2, the first year of using the MPT, Seth explained the impact that the tool had on him and his students:

The really big change was all that scaffolding that we built up to the proofs...which provided some of these comments [from students] like proofs were easy, you know, that was fun....I think back to my first year teaching proof. Straight agony....I probably, like when I took geometry, I sort of understood it myself...but I certainly didn't have a great grasp of how to teach it. I mean, as we said...I just threw it up one day, like here we go, we're gonna do a bunch of these and you have two options – you can either understand what's going on or you're gonna recognize that there's only about ten of them, like in different forms and you can probably, if you're good enough, you can memorize basically what's going on and survive. But...there's no takeaway from that. So I think the way we built it this year was remarkable in terms of their retention. (Seth, 6/5/12)

Like Seth, Mike also reported that he found that the tool supported him in teaching proof and supported his students in learning proof.

## DISCUSSION AND SUMMARY

The tool described in this paper was developed in response to some of the findings related to the challenges of teaching proof in high school geometry. The tool was intended to scaffold the introduction to proof for the students. In contrast to the traditional teaching methods reviewed in the literature, the tool was intended to assist the project teachers with introducing proof to their students in a manner that did not feel like such an “abrupt transition” (Moore, 1994) into the deep end of the proof pool. The five sub-goals of the tool were intended to provide teachers with a support for teaching proof. Although this study only presents findings from two teachers using the MPT, these findings are promising because the teachers did more than just use the tool

in a casual way. Rather, they saw enough potential in the tool use to develop a new curriculum around them, and they reported strong effects from their use. Additional research that explores the use of the Mathematical Proof Tool with additional teachers in varying contexts are warranted to determine if this tool can be used by teachers to improve the teaching and learning of proof, even potentially in other sub-areas of mathematics.

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